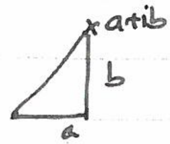
FP2 Complex Numbers

- $z = a + ib \qquad |z| = \sqrt{a^2+b^2} \qquad \arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$



  $z = r(\cos\theta + i\sin\theta)$

  $z = re^{i\theta}$

- $z^* = a - ib$

  $z^* = r(\cos\theta - i\sin\theta)$

  $z^* = re^{-i\theta}$

- De Moivres Thm: $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = re^{in\theta}$

  $z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta = re^{-in\theta}$

- $z + \frac{1}{z} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta$

  $z - \frac{1}{z} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2i\sin\theta$

  $z^n + \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n = 2\cos n\theta$

  $z^n - \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n = 2i\sin n\theta$

- $\left(z + \frac{1}{z}\right)^n = (2\cos\theta)^n = 2^n\cos^n\theta$

- $\left(z + \frac{1}{z}\right)^5 = z^5 + 5\frac{z^4}{z} + 10\frac{z^3}{z^2} + 10\frac{z^2}{z^3} + 5\frac{z}{z^4} + \frac{1}{z^5}$

  $= z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$

  $= \left[z^5 + \frac{1}{z^5}\right] + 5\left[z^3 + \frac{1}{z^3}\right] + 10\left[z + \frac{1}{z}\right]$

$\left[z^5 + \frac{1}{z^5}\right] = 2\cos 5\theta$

$\left[z^3 + \frac{1}{z^3}\right] = 2\cos 3\theta$

$\left[z + \frac{1}{z}\right] = 2\cos\theta$

$\therefore\ 2^5\cos^5\theta = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$

$= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$\therefore\ \cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$

$\left(z - \frac{1}{z}\right)^5 \Rightarrow \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$

- $C = \cos 2\theta - \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 8\theta - \frac{1}{8}\cos 11\theta \ldots$

  $S = \sin 2\theta - \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 8\theta - \frac{1}{8}\sin 11\theta \ldots$

  $\therefore C + iS = (\cos 2\theta + i\sin 2\theta) - \frac{1}{2}(\cos 5\theta + i\sin 5\theta) + \frac{1}{4}(\cos 8\theta + i\sin 8\theta) - \frac{1}{8}(\cos 11\theta + i\sin 11\theta) \ldots$

  $= e^{2\theta i} - \frac{1}{2}e^{5\theta i} + \frac{1}{4}e^{8\theta i} - \frac{1}{8}e^{11\theta i} \ldots$

  This is gp with $a = e^{2\theta i}$ $\quad r = -\frac{1}{2}e^{3\theta i}$

  Since we want a sum to infinity (look carefully at this

  $$S_\infty = C + iS = \frac{a}{(1-r)} = \frac{e^{2\theta i}}{1 + \frac{1}{2}e^{3\theta i}}$$

  To tidy the denominator, multiply as in rationalising

  $$\therefore C + iS = \frac{e^{2\theta i}\left(1 + \frac{1}{2}e^{-3\theta i}\right)}{\left(1 + \frac{1}{2}e^{3\theta i}\right)\left(1 + \frac{1}{2}e^{-3\theta i}\right)} \quad \text{and rearrange.}$$

  $$\begin{aligned}
  \left(1 + \tfrac{1}{2}e^{3\theta i}\right)\left(1 + \tfrac{1}{2}e^{-3\theta i}\right) &= 1 + \tfrac{1}{2}e^{3\theta i} + \tfrac{1}{2}e^{-3\theta i} + \tfrac{1}{4}e^{3i\theta}e^{-3i\theta} \\
  &= \tfrac{5}{4} + \tfrac{1}{2}\left(e^{3\theta i} + e^{-3\theta i}\right) \\
  &= \tfrac{5}{4} + \tfrac{1}{2}(2\cos 3\theta) \\
  &= \tfrac{5}{4} + \cos 3\theta
  \end{aligned}$$

  $$\begin{aligned}
  e^{2\theta i}\left(1 + \tfrac{1}{2}e^{-3\theta i}\right) &= e^{2\theta i} + \tfrac{1}{2}e^{-\theta i} \\
  &= \cos 2\theta + i\sin 2\theta + \tfrac{1}{2}\cos\theta - \tfrac{1}{2}i\sin\theta
  \end{aligned}$$

  $$\therefore C + iS = \frac{4\cos 2\theta + 4j\sin 2\theta + 2\cos\theta - 2j\sin\theta}{5 + 4\cos 3\theta}$$

  compare real parts $\quad C = \dfrac{4\cos 2\theta + 2\cos\theta}{5 + 4\cos 3\theta}$ etc.

- $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

  $e^{i\theta} - e^{-i\theta} = -2i\sin\theta$

- if $z = -2+2j$ $\quad |z| = \sqrt{8}$ $\quad \arg(z) = 3\pi/4$

  cube roots $\Rightarrow$ w say $\quad |w| = \sqrt[3]{\sqrt{8}} = \sqrt{2}$

  $\theta = \dfrac{\phi + 2k\pi}{n}$ $\quad k = 0, 1, 2,$ (cube roots) and $n = 3$ $\quad \phi = \arg(z) = 3\pi/4$

  $\theta = \frac{\pi}{4}, \frac{11\pi}{12}, \frac{-5\pi}{12}$ $\quad$ cube roots are $\sqrt{2}e^{\pi i/4}, \sqrt{2}e^{11\pi/12\, i}, \sqrt{2}e^{-\frac{5\pi i}{12}}$